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THE FRACTAL DIMENSION OF THE APOLLONIAN SPHERE PACKING

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Abstract

The fractal dimension of the Apollonian sphere packing has been computed numerically up to six trusty decimal digits. Based on the 31 944 875 541 924 spheres of radius greater than 2^{-19} contained in the Apollonian packing of the unit sphere, we obtained an estimate of 2.4739465, where the last digit is questionable. Two fundamentally different algorithms have been employed. Outlines of both algorithms are given.

1. INTRODUCTION

For the study of the classical *Apollonian packing*, it is convenient to treat the more general *osculatory packings*. In two dimensions, an osculatory packing of the unit disk S_0 is given by three pairwise externally tangent disks S_1, S_2, S_3 which are all internally tangent to S_0. More (open) disks S_4, ... are added in a way that each S_i is contained in S_0, disjoint from S_1, ... S_{i-1} and of maximal size. If S_1, S_2 and S_3 are chosen to be of equal size, the Apollonian packing is obtained (see Fig. 1). The disks S_0, S_1, S_2, S_3 are called the *initial circles* of the packing.

In the n-dimensional setting, the number of initial circles is $n + 2$. It is known that osculatory packings are *complete*, i.e. the sum of the area of S_1, S_2, ... converges to the area of S_0. Mandelbrot[1] defines the *fractal dimension D* of the packing as the Hausdorff-dimension of the residual set $S_0 \setminus \bigcup_{i=1}^{\infty} S_i$. A remarkable theorem due to Boyd[2] says that D is the same for all osculatory packings. It is generally believed that D cannot be found analytically.

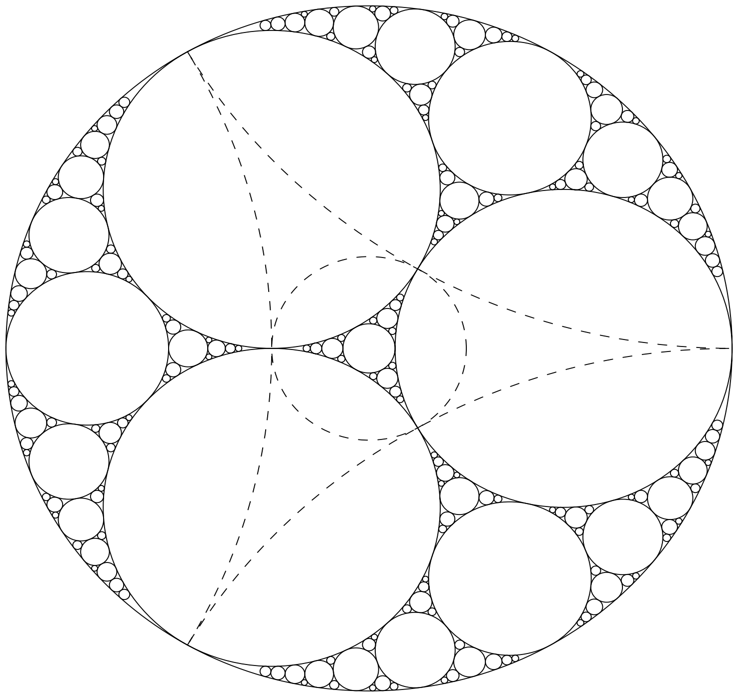

Figure 1: The 2-dimensional Apollonian packing. Dotted lines represent the inversion circles.

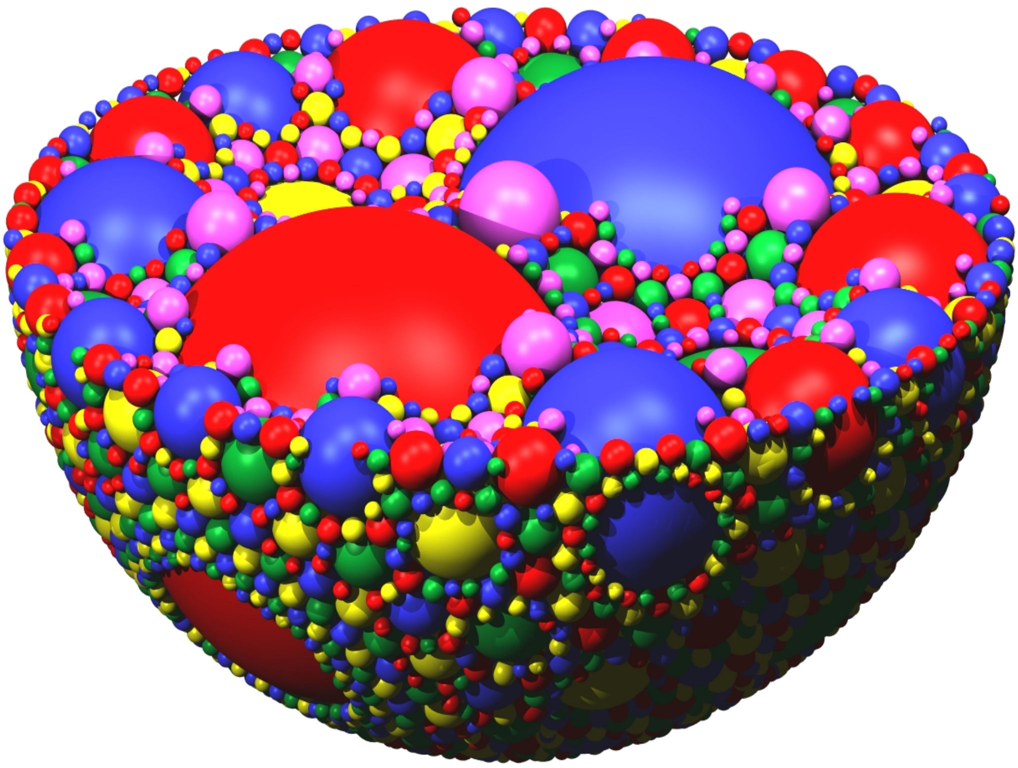

Figure 2: Part of the 3-dimensional Apollonian packing. For the meaning of the colors see Sec. 2.

Rigorous bounds are given by Melzak[3] and Boyd[4] for two dimensions and by Boyd and Larman[2] for three dimensions.

Numerical estimates can be obtained by restricting the packing to spheres of curvature (inverse radius) not exceeding a given value κ. Let $S(\kappa)$ denote this set, $N(\kappa)$ its cardinality, $s(\kappa)$ the sum of perimeters and $p(\kappa)$ the area of the set $S_0 \backslash \bigcup_{i=1}^{N(\kappa)-1} S_i$. Then, two theorems due to Boyd[5,6] imply the following asymptotic behavior of the three functions:

$$N(\kappa) \sim \kappa^{D} \qquad s(\kappa) \sim \kappa^{D-1} \qquad p(\kappa) \sim \kappa^{D-2} \tag{1}$$

Manna and Herrmann[7] used these expressions to obtain numerically the value for the fractal dimension of 1.30568... in two dimensions. They remark that the values based on $s(\kappa)$ converge slower. This is, however, just the effect of not having included the circle S_0.

Boyd was the first to attack numerically the *3-dimensional* problem (see Fig. 2). In Ref 8, he conjectures a value of approx. 2.42. It is our significantly different result which motivated us to write this note.

Although the definition of osculatory packings is *constructive*, an efficient algorithm cannot be derived from it. Instead, two fast algorithms are sketched in the next two sections.

2. THE "INVERSION ALGORITHM"

As Mandelbrot points out in Ref. 1, 2-dimensional osculatory packings can be constructed by repeatedly applying *circular inversion* to the initial circles. A set of four inversion circles is sufficient, namely those keeping fixed three out of four initial circles (see Fig. 1).

We notice that every packing circle which is not an initial circle lies completely inside one and outside all other inversion circles. Therefore, it has exactly one larger image. This observation allows to construct a simple algorithm for generating each circle exactly once: Starting with the initial circles, apply to each circle exactly those of the four inversions that reduce its size. If only circles of a certain minimum size are to be generated, this process can be made finite by cutting each computation branch whenever a smaller circle has been produced. Finally, by assigning distinct *colors* to the initial circles, the packing can be divided into four *clans*, i.e. subsets of constant color (as shown in Fig. 2).

In three dimensions, inversion spheres overlap so that some of the packing spheres have *two* larger images. The naïve adaptation of the above algorithm would therefore generate all spheres, however some of them more than once (but with consistent colors). To restore the uniqueness we allow not all size-reducing inversions, but only those which produce spheres having their center in the "target region" of the respective inversion sphere. The target regions are the insides of the inversion spheres minus parts of the regions of overlap. They are chosen such that their disjoint union equals the union of the inversion spheres. It is this simple algorithm that we have used to obtain numerically the fractal dimension of the three-dimensional Apollonian packing.

As a matter of efficiency, it is advisable to represent spheres not with Cartesian coordinates x_1 ,..., x_n and a radius r but with *inversive coordinates*

$$\begin{aligned} a_1 &= \frac{x_1}{r}, \ldots, a_n = \frac{x_1}{r} \\ a_{n+1} &= \frac{x_1^2 + \ldots + x_n^2 - r^2 - 1}{2r} \\ a_{n+2} &= \frac{x_1^2 + \ldots + x_n^2 - r^2 + 1}{2r} \end{aligned} \tag{2}$$

In these coordinates, the inversion becomes a *linear* transformation of the coordinate vector. Note that also the curvature can be expressed linearly (namely $a_{n+2} - a_{n+1}$), and so can the criterium for the target regions.

To further simplify calculations, coordinates can be altered again by the linear transformation

$$T = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \frac{\sqrt{6}}{12} & \frac{\sqrt{6}}{12} & \frac{\sqrt{6}}{12} & \frac{\sqrt{6}}{12} & 0 \end{bmatrix} \tag{3}$$

leading to *integer* coordinates for the initial spheres

$$\begin{gathered} \langle 0, 2, 2, 2, 1 \rangle, \qquad \langle 2, 0, 2, 2, 1 \rangle, \\ \langle 2, 2, 0, 2, 1 \rangle, \qquad \langle 2, 2, 2, 0, 1 \rangle, \\ \langle 0, 0, 0, 0, -1 \rangle, \end{gathered} \tag{4}$$

and simpler linear mappings for the spherical inversions:

$$I_1 = \begin{bmatrix} -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ldots,$$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (5)$$

$$I_5 = \begin{bmatrix} 2 & 1 & 1 & 1 & -6 \\ 1 & 2 & 1 & 1 & -6 \\ 1 & 1 & 2 & 1 & -6 \\ 1 & 1 & 1 & 2 & -6 \\ 1 & 1 & 1 & 1 & -5 \end{bmatrix}$$

3. THE “SODDY ALGORITHM”

In n-space, the maximum number of mutually tangent spheres is $n + 2$, if one demands that no three spheres have a common point. We will call such a set of spheres a *Soddy* set. Soddy's formula[9] is an identity for the curvatures in a Soddy set:

$$n \sum_{i=1}^{n+2} \kappa_i^2 = \left(\sum_{i=1}^{n+2} \kappa_i \right)^2 \qquad (6)$$

If it is solved for one of the κ_i, we get exactly two distinct solutions. Hence, it is always possible to replace any sphere in a Soddy set by its “second solution”, the result is again a Soddy set. We might call this a *Soddy operation*. Since the *sum* of the solutions of Eq. (6) is:

$$\frac{2}{n-1} \sum_{j=1, j \neq i}^{n+2} \kappa_j \qquad (7)$$

Soddy operations can be computed as *linear* functions on $n + 2$-vectors (assuming that coordinates are of no interest).

Applying these definitions to osculatory packings, we notice that the set of initial spheres is a Soddy set. A method to produce all spheres of the packing is to apply to the set of initial spheres all $n + 2$ Soddy operations and to all its successors those $n + 1$ which do not invert the last operation.

In two dimensions, this procedure has two properties:

(i) Each Soddy operation generates a circle which is smaller than the four circles in the Soddy set.

(ii) The same circle will not be generated twice. If the goal is to enumerate the circles of a certain *minimum size* only, we can cut each branch of the computation as soon as a smaller circle has been generated. This algorithm has been used by Melzak[10].

In three dimensions, neither (i) nor (ii) hold. Based on geometrical evidence, we found a modification to enforce (i) and (ii). Unlike Boyd’s algorithm[8], it remains coordinate-free. We will make use of the *coloring* introduced in the last section. One can prove by induction that the spheres in a Soddy set are always of distinct colors.

The modification basically consists in cutting additional computation branches. The cases to be treated specially are:

- If the newly generated sphere S_g has larger radius than the smallest sphere S_s in the Soddy set, cut the branch.
- If $rad(S_g) = rad(S_s)$, do not count the new sphere, but cut the branch only if $color(S_g) < color(S_s)$. This is to cut exactly one of two identical branches. The current Soddy set will be produced in a second branch with the rôles of S_g and S_s swapped.
- If $rad(S_g) < rad(S_s)$, do not cut the branch. Count the new sphere in general. In order to find the exceptions, perform a “look-ahead” Soddy operation replacing S_s and producing a new sphere S_l.
  - If $rad(S_l) > rad(S_s)$, do not count the sphere.
  - If $rad(S_l) = rad(S_s)$, count the new sphere only as a half.

Although we do not have a rigorous correctness proof, this 3-dimensional Soddy-algorithm has been helpful for double-checking the results of the inversion algorithm.

4. NUMERICAL RESULTS

We generated all spheres of curvature less than 2^{19} calculating on a cluster of 20 DEC 3000-400AXP workstations during 5.5 days. The spheres have been used to sample the three functions $N(\kappa)$, $s(\kappa)$ and $p(\kappa)$ at the values $\kappa = 2^{0.1}, 2^{0.2}, \ldots, 2^{19.0}$.

Table 1: Excerpt from the Computer Output

$log_2(\kappa)$	$N(\kappa)$	$s(\kappa)$	$p(\kappa)$
2	9	26.94641279	2.258953709774
5	1236	76.96194118	0.717030382959
10	6335400	396.34565660	0.116244349769
15	33532722660	2048.53328254	0.018774688805
19	31944875541924	7623.11018362	0.004366569649

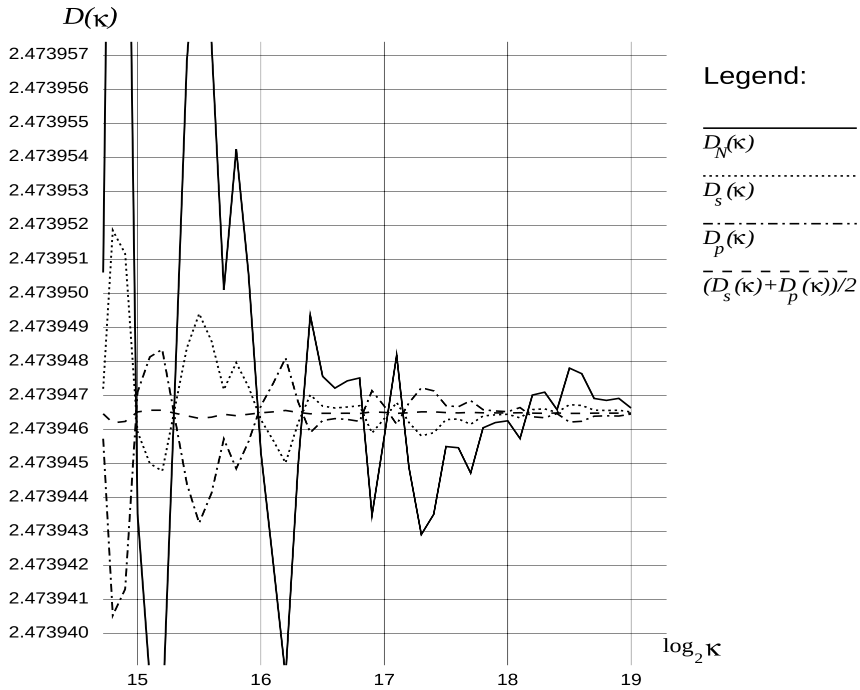

Figure 3: Estimates for the fractal dimension as a function of the log_2 of the maximal curvature.

Table 1 displays them for some values of κ. We also confirm Boyd's number (see Ref. 8) of 1 693 595 spheres having a curvature less than 601.

For each ten successive samples we did a least-square-fit using the asymptotic Eq. (1) in logarithmic form. The obtained estimates for D which we denote by $D_N(\kappa)$, $D_s(\kappa)$, $D_p(\kappa)$ are plotted in Fig. 3. It is interesting to observe that the fluctuations of the latter two almost cancel each other. Their arithmetic mean is also shown in Fig. 3 and yields the estimate of 2.4739465 ± 0.0000001 for the fractal dimension of the three-dimensional osculatory packing.

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